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1995 J. Phys. A: Math. Gen. 28 4251

(http://iopscience.iop.org/0305-4470/28/15/006)

J. Phys. A: Math. Gen. **28** (1995) 4251–4259. Printed in the UK

A semiquantum Fokker–Planck equation for spin–velocity relaxation of gas particles

L V Il'ichov

Institute of Automation and Electrometry SB RAS, Novosibirsk State University, Novosibirsk, 630090, Russia

Abstract. A model Fokker–Planck kinetic equation is postulated which describes coupled spin–velocity relaxation of test particles immersed in a buffer gas. The proposed model accounts for specific parts of the friction force ('Magnus'-type and 'sailing'-type) to which a spinning test particle is subjected. Special attention is paid to the 'Magnus phenomenon' in the case of spin-half particles. Some curious thermodynamic aspects are discussed.

1. Introduction

In an earlier work (Il'ichov 1991a) the following Fokker–Planck equation was proposed:

$$\partial_t \rho(\boldsymbol{v}, \boldsymbol{s}, t) = \nu_0 \nabla \cdot (\boldsymbol{v} + \bar{v}^2 \nabla)\, \rho(\boldsymbol{v}, \boldsymbol{s}, t) + \nu_1 \nabla \cdot (\boldsymbol{v} \times \boldsymbol{s})\, \rho(\boldsymbol{v}, \boldsymbol{s}, t) + \nu_2 (\boldsymbol{s} \cdot \nabla) \left[(\boldsymbol{v} \cdot \boldsymbol{s}) + \bar{v}^2 (\boldsymbol{s} \cdot \nabla) \right] \rho(\boldsymbol{v}, \boldsymbol{s}, t) \quad (1)$$

where $\bar{v}$ is the mean thermal velocity and ∇ the symbol of the velocity derivative. It was expected to describe the velocity relaxation under some conditions of heavy spinning particles immersed in an atmosphere of a light buffer gas. The essential properties of the model are as follows. The test particles carry semiclassical internal angular momentum (spin) $\boldsymbol{J} = J\boldsymbol{s}$ which has quantized magnitude J but continuous (classical) orientation $\boldsymbol{s}$ (Nasirov and Shalagin 1981). Collisions are assumed to change neither the spin value J nor its direction $\boldsymbol{s}$. In other words the test particles are heavy rapidly rotating tops. In addition, J is assumed to be the same for all test particles. As a result the subensemble of test particles, specified by $\boldsymbol{s}$, relaxes independently of particles with other spin directions. Being unaffected by collisions, the spin direction modifies the character of the velocity relaxation. This is reflected in the second line of (1). The term $\propto \nu_1$ accounts for phenomenon related to the classical Magnus effect, the deviation of the trajectory of a rotating flying body due to the Magnus force $\boldsymbol{F}_M \propto (\boldsymbol{V} \times \Omega)$, where $\boldsymbol{V}$ is the velocity of the body, and Ω is its angular velocity. The process behind the term $\propto \nu_2$ accounts for the various mobilities of an aligned particle along the alignment axis and in the transverse direction. This phenomenon has a classical counterpart in sailing motion. The quantum 'Magnus phenomenon' and the 'sailing phenomenon' firstly followed from the analysis of gas dynamic equations derived from the Waldmann–Snider quantum kinetic equation (Gel'mukhanov and Il'ichov 1985).

In equation (1) these phenomena reveal themselves at the classical kinetic level. Equation (1) is a generalization of the well known model of weak collisions—the first line in (1) (see, e.g., McCourt *et al* 1990). As one can easily verify, this generalization is unique if we allow the drift term of the generalized equation to be linear with respect to $\boldsymbol{v}$ and the diffusion term be independent of velocity. Strong reasons for these constraints were presented by Rautian (1991).

0305-4470/95/154251+09$19.50

As we have already stressed, the model, based on (1), considers the test gas as a mixture of independent subcomponents specified by $s \in S^2$. This limitation is a sequence of the semiclassical approach to angular momentum. In reality any change of velocity in the course of a collision is accompanied by some spin deorientation. In the present paper we are going to propose a quantum modification of (1), which accounts for this deorientation. We have no reason to revise the classical nature of translational motion. All modifications will concern spin which should be considered in a purely quantum way. The v-dependent spin density matrix $\hat{\rho}(v, t)$ will take the place of $\rho(v, s, t)$ in the proposed modification of (1). This is the main distinctive property of our model in comparison with others based on a purely quantum line of attack (Haken 1969, Morozov 1981). Starting from known quantum kinetic equations, these authors derived generalized Fokker–Planck equations for c-number representatives of statistical operators. We are going to move in another direction—from the classical equation (1) to a semi-quantum equation using the correspondence principle. In this point our approach is in some respects closer to that of Schramm *et al* (1985), where a classical Langevin equation is extended to the quantum domain through a modification of the guiding stochastic process. This method seems the most applicable to translational motion, whereas we deal with the coupling of classical translational motion and quantum rotation.

2. Semiquantum Fokker–Planck equation

The main object of the present paper will be the following generalization of (1):

$$\partial_t \hat{\rho}(v, t) = \nabla_i \left\{ v_j \mathcal{A}_{ij} \left[\hat{\rho}(v, t) \right] + \bar{v}^2 \nabla_j \mathcal{B}_{ij} \left[\hat{\rho}(v, t) \right] \right\} + \text{(non-deriv. terms)} \tag{2}$$

where the velocity distribution $\hat{\rho}(v, t)$ is at the same time a statistical operator in the spin space of the test particles; $\mathcal{A}_{ij}$ and $\mathcal{B}_{ij}$ are superoperators in spin space and at the same time are tensors in velocity space. The non-derivative terms will be discussed later.

$\mathcal{B}_{ij}$ is evidently symmetric tensor:

$$\mathcal{B}_{ij} = \mathcal{B}_{ji} \tag{3}$$

whereas $\mathcal{A}_{ji}$ may have both symmetric and antisymmetric parts

$$\begin{aligned} \mathcal{A}_{ij} &= \mathcal{A}^{s}_{ij} + \mathcal{A}^{a}_{ij} \\ \mathcal{A}^{s}_{ij} &= \mathcal{A}^{s}_{ji} \qquad \mathcal{A}^{a}_{ij} = -\mathcal{A}^{a}_{ji}. \end{aligned} \tag{4}$$

Keeping in mind the correlation between equations (1) and (2), we expect the 'Magnus phenomenon' to be generated by $\mathcal{A}^{a}_{ij}$ and the 'sailing phenomenon' by $\mathcal{A}^{s}_{ij}$.

There is a relation between $\mathcal{A}^{s}_{ij}$ and $\mathcal{B}_{ij}$ which follows from the equilibrium condition. We assume the derivative as well as the non-derivative operators give zero when acting on the equilibrium distribution $\hat{\rho}_{eq}(v) = \hat{\rho}_{eq} W(v)$, where $W(v)$ is the Maxwellian distribution in velocity space; the operator $\hat{\rho}_{eq}$ is proportional to the identity operator $\hat{E}^{(J)}$ in every J-spin subspace. Substituting $\hat{\rho}_{eq}(v)$ in (2), we arrive at the following relation:

$$\mathcal{A}^{s}_{ij}[\hat{\rho}_{eq}] = \mathcal{B}_{ij}[\hat{\rho}_{eq}]. \tag{5}$$

We postulate the simplest form of $\mathcal{A}_{ij}$ and $\mathcal{B}_{ij}$ which allows the reduction of (2) to (1) in the classical limit and preserves the Hermitian character of the statistical operator:

$$\begin{aligned}\mathcal{A}_{ij}^{(s)}[\hat\rho] &= a_0\delta_{ij}\hat\rho + \tfrac{1}{2}a_2^{(1)}\left\{\left(\hat J_i\hat J_j + \hat J_j\hat J_i\right)\hat\rho + \hat\rho\left(\hat J_j\hat J_i + \hat J_i\hat J_j\right)\right\}\\ &\quad + a_2^{(2)}\left(\hat J_i\hat\rho\hat J_j + \hat J_j\hat\rho\hat J_i\right)\\ \mathcal{A}_{ij}^{(a)}[\hat\rho] &= \varepsilon_{ijk}\left(a_1\hat J_k\hat\rho + a_1^*\hat\rho\hat J_k\right) + \mathrm{i}a_2^{(3)}\left(\hat J_i\hat\rho\hat J_j - \hat J_j\hat\rho\hat J_i\right)\\ \mathcal{B}_{ij}^{(s)}[\hat\rho] &= b_0\delta_{ij}\hat\rho + \tfrac{1}{2}b_2^{(1)}\left\{\left(\hat J_i\hat J_j + \hat J_j\hat J_i\right)\hat\rho + \hat\rho\left(\hat J_j\hat J_i + \hat J_i\hat J_j\right)\right\}\\ &\quad + b_2^{(2)}\left(\hat J_i\hat\rho\hat J_j + \hat J_j\hat\rho\hat J_i\right)\end{aligned} \tag{6}$$

where a_0, $a_2^{(1)}$, $a_2^{(2)}$, $a_2^{(3)}$, b_0, $b_2^{(1)}$, $b_2^{(2)}$ are real quantities and $a_1 = a_1' + \mathrm{i}a_1''$ is complex. All these quantities have a phenomenological nature in our context. It follows from (5) that in every J-spin subspace

$$(a_0 - b_0)\hat E^{(J)}\delta_{ij} = \left(b_2^{(1)} + b_2^{(2)} - a_2^{(1)} - a_2^{(2)}\right)\left(\hat J_i\hat J_j + \hat J_j\hat J_i\right). \tag{7}$$

Note that we use only the terms of the first and the second degree with respect to the angular momentum operators $\hat J_i$ $(i = 1, 2, 3)$. The simplest non-derivative term of the same kind is

$$\gamma\left(2\hat{\boldsymbol J}\,\hat\rho\,\hat{\boldsymbol J} - \hat J^2\hat\rho - \hat\rho\hat J^2\right)/2. \tag{8}$$

In the context of our model no change of J takes place. The non-derivative term describes isotropic destruction of the coherence between Zeemann sublevels due to collisions. In the derivative term (equation (2)) the relaxation of spin orientation is unseparable from the velocity relaxation, because the result of a collision depends on the relative orientation of $\boldsymbol v - \boldsymbol J$ (Kuščer *et al* 1981). The superoperator term, $\mathcal{A}_{ij}[\hat\rho(\boldsymbol v, t)]v_j$, plays the role of the ith component of a friction force $\boldsymbol F_{\mathrm{fr}}(\boldsymbol v)$ which a test particle with velocity $\boldsymbol v$ is subjected to. To be more precise we write

$$\boldsymbol F_{\mathrm{fr}}(\boldsymbol v)\,\mathrm{Tr}\,\hat\rho(\boldsymbol v, t) = \mathrm{Tr}\left(\boldsymbol e_i\,\mathcal{A}_{ij}[\hat\rho(\boldsymbol v, t)]\,v_j\right) \tag{9}$$

where $\boldsymbol e_i$ $(i = 1, 2, 3)$ are the unit orthonormal vectors of the laboratory coordinate system. After cyclic permutation of operator factors under the trace the superoperator $\mathcal{A}_{ij}$ can be written as the friction force operator which was introduced by Bloemink *et al* (1994). An equation similar to (9) relates $\mathcal{B}_{ij}[\hat\rho(\boldsymbol v, t)]$ to an anisotropic diffusion coefficient in the velocity space.

Let us compare equation (2) where $\mathcal{A}_{ij}$ and $\mathcal{B}_{ij}$ are determined by (6) with the classical equation (1). There are no problems with identification of terms responsible for the 'sailing phenomenon'; these terms are proportional to $a_2^{(1)}$ and $a_2^{(2)}$. A different situation arises with the 'Magnus phenomenon'. Note that only the term $\propto a_1'$ in (6) is akin to the corresponding classical term in (1), whereas the terms $\propto a_1''$ and $a_2^{(3)}$ have no classical counterparts because they are conditioned by non-commutativity of operators. Further still, we should not consider the term $\propto a_2^{(3)}$ to be related to the 'Magnus'-type force. This term is bilinear in $\boldsymbol J$, but it is evident from common physical reasons that the 'Magnus phenomenon' must be odd with respect to $\boldsymbol J$. The uncertain status of the term $\propto a_2^{(3)}$ will be clarified in the next section by the simpler example of spin-half particles.

3. The case of spin-half particles

Let us consider the case of spin-half test particles. Their statistical operator is as follows:

$$\hat\rho(\boldsymbol v, t) = \left(\rho_0(\boldsymbol v, t) + \rho_k(\boldsymbol v, t)\,\hat\sigma_k\right)/2 \tag{10}$$

where $\hat{\sigma}_k$ $(k = 1, 2, 3)$ are the Pauli matrices, and $\rho_0(v, t)$ and $\rho_k(v, t)$ are the velocity distribution of the test particles and the distribution of the kth component of their orientation (mean spin vector), respectively.

The model of spin-half particles can describe the 'Magnus phenomenon', but not the 'sailing phenomenon'. The latter is conditioned by alignment (by a second rank tensor with respect to spin variables), but alignment does not exist in spin-half system.

By equations (2), (6) we have

$$\begin{aligned}
\partial_t \rho_0(v, t) = {} & (a_0 + a_2^{(1)}/2 + a_2^{(2)}/2)\nabla \cdot v\, \rho_0(v, t) \\
& + (b_0 + b_2^{(1)}/2 + b_2^{(2)}/2)\, \bar{v}^2 \nabla \cdot \nabla \rho_0(v, t) \\
& + (a_1' + a_2^{(3)}/2)(\nabla \times v) \cdot \rho\, (v, t) \\
\partial_t \rho\, (v, t) = {} & (a_0 + a_2^{(1)}/2 - a_2^{(2)}/2)(\nabla \cdot v)\, \rho\, (v, t) \\
& + (b_0 + b_2^{(1)}/2 - b_2^{(2)}/2)\, \bar{v}^2 (\nabla \cdot \nabla)\, \rho\, (v, t) \\
& + (a_1' - a_2^{(3)}/2)(\nabla \times v)\, \rho_0(v, t) \\
& + (a_2^{(2)}/2 - a_1'')\, [\, \rho\, (v, t) + v\, (\nabla \cdot \rho\, (v, t))\,] \\
& + (a_2^{(2)}/2 + a_1'')\nabla (v \cdot \rho\, (v, t)) \\
& + \bar{v}^2 b_2^{(2)} \nabla (\nabla \cdot \rho\, (v, t)) - \gamma \rho\, (v, t).
\end{aligned} \tag{11}$$

Relation (7) is replaced by

$$a_0 + a_2^{(1)}/2 + a_2^{(2)}/2 = b_0 + b_2^{(1)}/2 + b_2^{(2)}/2. \tag{12}$$

This equation states the well known identity of transport collision rate (the LHS of (12)) to that appearing in the velocity space diffusion coefficient. Now we are going to compare the quantum 'Magnus phenomenon' with its classical counterpart. The term $\propto \nu_1$ in (1) accounts for it. Using equation (1), we derive the following equation for the gas-dynamic flux with definite spin direction:

$$\partial_t\, j(s, t) = -\nu_0 j(s, t) + \nu_1 (s \times j(s, t)) - \nu_2 s\, (s \cdot j(s, t)) \tag{13}$$

where

$$j(s, t) \equiv \int v\, \rho(v, s, t)\, \mathrm{d}^3 v. \tag{14}$$

We see that the 'Magnus phenomenon' results in the precession of $j(s, t)$ around s with the angular velocity ν_1. As was shown by Gel'mukhanov and Il'ichov (1985) and by Il'ichov (1990b) the quantum 'Magnus phenomenon' manifested itself as a collisional coupling of the ordinary flux of particles $j_0(t)$:

$$j_0(t) \equiv \int v\, \rho_0(v, t)\, d^3 v \tag{15}$$

and the flux of orientation $j_1(t)$:

$$j_1(t) \equiv \int (v \times \rho(v, t))\ \mathrm{d}^3 v. \tag{16}$$

From (11) we get

$$\begin{aligned}
\partial_t\, j_0(t) &= -\nu(00)\, j_0(t) + \nu(01)\, j_1(t) \\
\partial_t\, j_1(t) &= -(\nu(11) + \gamma)\, j_1(t) + \nu(10)\, j_0(t)
\end{aligned} \tag{17}$$

where

$$\begin{aligned}\nu(00) &\equiv a_0 + a_2^{(1)}/2 + a_2^{(2)}/2 \\ \nu(11) &\equiv a_0 + a_2^{(1)}/2 - a_2^{(2)} + a_1'' \\ \nu(01) &\equiv a_1' + a_2^{(3)}/2 \\ \nu(10) &\equiv -2(a_1' - a_2^{(3)}/2).\end{aligned} \tag{18}$$

We see that $a_2^{(3)}$ appears along with a_1' in the collision rates $\nu(01)$ and $\nu(10)$ responsible for the coupling. This is rather strange. To clarify the situation one should invoke the Onsager–Casimir reciprocal relations (Il'ichov 1990b) which, on being applied to the 'Magnus phenomenon', give $\nu(10) = -2\nu(01)$. Therefore $a_2^{(3)} = 0$ and the corresponding term in (6) must be eliminated.

We see from (18) that if $2a_1'' \neq a_2^{(2)}$ the transport collision rate $\nu(11)$ for oriented particles is not equal to the ordinary transport collision rate $\nu(00)$. For simplicity we assume $a_2^{(2)} = b_2^{(2)} = a_2'' = 0$ and arrive at the following equations:

$$\begin{aligned}\partial_t \rho_0(\boldsymbol{v}, t) &= \nu_0 \nabla \cdot (\boldsymbol{v} + \bar{v}^2 \nabla) \rho_0(\boldsymbol{v}, t) + \nu_1 (\nabla \times \boldsymbol{v}) \cdot \boldsymbol{\rho}(\boldsymbol{v}, t) \\ \partial_t \boldsymbol{\rho}(\boldsymbol{v}, t) &= \nu_0 \left[\nabla \cdot (\boldsymbol{v} + \bar{v}^2 \nabla)\right] \boldsymbol{\rho}(\boldsymbol{v}, t) + \nu_1 (\nabla \times \boldsymbol{v}) \rho_0(\boldsymbol{v}, t) - \gamma\, \boldsymbol{\rho}(\boldsymbol{v}, t)\end{aligned} \tag{19}$$

where $\nu_0 \equiv a_0 + a_2^{(2)}$, $\nu_1 \equiv a_1'$; ν_1 from (19) should not be confused with ν_1 from (1). The terms $\propto \nu_0$ on the right-hand side of (15) describe an irreversible velocity relaxation (an Ornstein–Uhlenbeck process). The term with γ is responsible for the collisional decay of the orientation and has irreversible nature too. At the same time, in spite of their collisional origin, the terms $\propto \nu_1$ in (15) describe the reversible Magnus process which is invariant under the transformation

$$\begin{aligned}\rho_0(\boldsymbol{v}, t) &\to \rho_0(-\boldsymbol{v}, -t) \\ \boldsymbol{\rho}(\boldsymbol{v}, t) &\to -\boldsymbol{\rho}(-\boldsymbol{v}, -t).\end{aligned} \tag{20}$$

The main advantage of the differential equations (19) compared with the more rigorous model based on integro-differential Waldman–Snider equation (Gel'mukhanov and Il'ichov 1985) is their solvability. The solution to (19) is presented in the appendix.

4. Does the quantum 'Magnus phenomenon' 'violate' the second law of thermodynamics?

We are going to propose a suitable Lyapunov functional for the system (19). We consider the following functional:

$$H(t) = \int W^{-1}(\boldsymbol{v}) \left(\rho_0^2(\boldsymbol{v}, t) + \rho^2(\boldsymbol{v}, t)\right)\, \mathrm{d}^3 v \tag{21}$$

where $\rho(\boldsymbol{v}, t)$ is the length of the orientation vector $\boldsymbol{\rho}(\boldsymbol{v}, t)$. It is easy to show that the time derivative $H(t)$ is negative:

$$\dot{H}(t) = -2\nu_0 \int W^{-1}(\boldsymbol{v}) \bigg[\left(\bar{v} \nabla \rho_0(\boldsymbol{v}, t) + \frac{\boldsymbol{v}}{\bar{v}} \rho_0(\boldsymbol{v}, t)\right)^2 + \sum_{i,k} \left(\bar{v} \nabla_i \rho_k(\boldsymbol{v}, t) + \frac{v_i}{\bar{v}} \rho_k(\boldsymbol{v}, t)\right)^2\bigg]\, \mathrm{d}^3 v - 2\gamma \int W^{-1}(\boldsymbol{v})\, \rho^2(\boldsymbol{v}, t)\, \mathrm{d}^3 v. \tag{22}$$

The well known argument (van Kampen 1984) proves that the evolution described by (19) tends to equilibrium: $\rho_0(v,t) \to W(v)$, $\rho(v,t) \to 0$.

It is significant that the 'Magnus phenomenon' makes no contribution to $H(t)$. The idea that no reversible process should contribute to any sensible H-function determined the choice of the functional (21).

One naturally wonders if the H-function (21) is unique for the system (15). We can try the free energy of the test gas (its definition being due to von Neumann (1955))

$$F(t) = T \int \mathrm{Tr}\,\hat{\rho}(v,t)\,\ln[\,\hat{\rho}(v,t)\,/\,W(v)\,]\,\mathrm{d}^3v \tag{23}$$

where T is the buffer gas temperature. In (23) we assume

$$\int \mathrm{Tr}\,\hat{\rho}(v,t)\,\mathrm{d}^3v = 1. \tag{24}$$

Using the basis of spin state vectors which diagonalized the statistical operator $\hat{\rho}(v,t)$ (this basis may depend on the velocity), one can prove that $F(t) \geqslant 0$. In the case of spin-half particles we have (up to an additive constant):

$$F(t)/T = \int \left[\frac{\rho_0(v,t)+\rho(v,t)}{2} \ln \frac{\rho_0(v,t)+\rho(v,t)}{W(v)} + \frac{\rho_0(v,t)-\rho(v,t)}{2} \ln \frac{\rho_0(v,t)-\rho(v,t)}{W(v)} \right] \mathrm{d}^3v. \tag{25}$$

The time derivative $\dot{F}(t)$ has the following components:

$$\dot{F}(t) = [\dot{F}(t)]_{\mathrm{fr}} + [\dot{F}(t)]_{\mathrm{diff}} + [\dot{F}(t)]_{\mathrm{dec}} + [\dot{F}(t)]_{\mathrm{Magn}}. \tag{26}$$

As was shown by Gardiner (1985), the first term in (26) (the friction term $\propto \nu_0$) is zero, and the second (the diffusional term $\propto \nu_0$) and third (the decay term $\propto \gamma$) are negative. After some manipulation we arrive at

$$[\dot{F}(t)]_{\mathrm{Magn}} = \frac{\nu_1}{2} T \int \left[\rho_0(v,t) \ln \frac{\rho_0(v,t)+\rho(v,t)}{\rho_0(v,t)-\rho(v,t)} - 2\rho(v,t) \right] v \cdot [\nabla \times n(v,t)]\ \mathrm{d}^3v \tag{27}$$

where $n(v,t) = \rho\,(v,t)/\rho(v,t)$. The term in the square brackets is positive and the sign of the expression (27) is determined by the term $v \cdot [\,\nabla \times n(v,t)]$. By preparing the state of the gas in a proper way we are able to make the integral (27) positive. As has been shown above, the test gas will definitely come to equilibrium. However, the dissipation of the free energy of the test gas is not to be monotonous. If $[\dot{F}(t)]_{\mathrm{Magn}}$ is positive and dominates, the total time derivative $\dot{F}(t)$ will also be positive at an initial stage of evolution. The specific gas states which demonstrate such behaviour are conditioned by a non-trivial velocity dependence of the orientation vector.

5. Conclusion

We have proposed a model which describes coupled spin–velocity relaxation of a test gas in an atmosphere of a buffer gas. In spite of the phenomenological character of our approach, it seems capable of accounting for the 'Magnus' and 'sailing' phenomena which find their origin in the rigorous model (Gel'mukhanov and Il'ichov 1985). Equations (2) and (8) exhibit a similarity with those arising in the theory of quantum Markovian processes (Gardiner 1985).

The 'Magnus phenomenon' for spin-half particles was considered in more detail. It turns out that not all proposed phenomenological terms in A_{ij} are allowed because of constraints imposed by the Onsager–Casimir reciprocal relations. Equations (19) admit a time-dependent solution, which can be used in the construction of a new kinetic model (see the appendix).

The result of section 4 contrasts with the conventional H-theorem. The situation suggests two explanations: (i) the proposed model is pathological, or (ii) a defect is contained in the definition of $F(t)$ (23). One may prefer the first alternative because the model described was postulated, and no systematic rigorous derivation was presented. Nevertheless, we should note that the derivation of equations (15) is possible. It starts from the Waldmann–Snider kinetic equation using the method similar to that of Kramers and Moyal (Moyal 1949). This fact supports the hope that the general quantum Fokker–Planck equation (2) for higher spin values is also a useful model. This derivation and some possibilities for the second alternative will be discussed elsewhere. We considered the relaxation of spin orientation in the framework of the $su(2)$ algebra of angular momentum. In principle, extending $su(2)$ to the Lie algebra of the metaplectic group $Mp(4,\mathrm{R}) \supset SU(2)$, one can account for the inelastic J–J transitions, as will also be shown elsewhere.

Acknowledgments

The author is deeply indebted to Professors S G Rautian and A M Shalagin, and especially to Professor I Kuščer for very helpful discussions. Support from the Russian Foundation for Fundamental Research (grant No 93-02-03567), the Netherlands Organization for Scientific Research, and the International Science Foundation (grant RCM000) is gratefully acknowledged.

Appendix

It is useful to introduce the conditional probability

$$Q(v, s, t|v_0, s_0, 0) = \frac{1}{2\pi}\langle s\,|\hat{\rho}(v,t)|\,s\,\rangle \qquad \text{(A1)}$$

for a test particle (which was initially prepared with velocity v_0 and in the spin state $|s_0\rangle$) to have at the time t the velocity v and to pass through a spin analyser with the axis along s. The expression (A1) is the well known Q-symbol of the statistical operator in the representation of coherent states of the group $SU(2)$ (Perelomov 1985). In order to make the formulae less cumbersome, we evaluate the function Q for the case $\gamma = 0$. Under this condition one can prove that the reversible factor of the evolution of $\hat{\rho}(v,t)$ can be separated from the irreversible factor:

$$Q(v, s, t|v_0, s_0, 0) = \int P_{\mathrm{OU}}(v, t|v', 0)\, P_{\mathrm{M}}(v', s, t|v_0, s_0, 0)\, \mathrm{d}^3v'. \qquad \text{(A2)}$$

In this formula $P_{\mathrm{OU}}(v, t|v', 0)$ is the transition probability (propagator) for the ordinary Ornstein–Uhlenbeck process

$$P_{\mathrm{OU}}(v, t|v', 0) = [\pi\sigma(t)]^{-3/2} \exp\left[-\frac{[v - v'\exp(-\nu_0 t)]^2}{\sigma(t)}\right] \qquad \text{(A3)}$$

where $\sigma(t) = 2\bar{v}^2[1 - \exp(-2\nu_0 t)]$. The propagator $P_M(v', s, t|v_0, s_0, 0)$ for the reversible 'Magnus process' has the following form:

$$P_M(v', s, t|v_0, s_0, 0) = \frac{1}{4\pi}\exp\left[\frac{v_0^2 - v'^2}{4\bar{v}^2}\right]\left[p_0(v', t|v_0, s_0, 0) + s\cdot p(v', t|v_0, s_0, 0)\right] \tag{A4}$$

where

$$\begin{aligned} p_0(v', t|v_0, s_0, 0) = \sum_{n,l,m}\langle v'|nlm\rangle \Big\{&\langle nlm|v_0\rangle \cos\omega_l t \\ &+\frac{\mathrm{i}\nu_1}{2\omega_l}\sqrt{(l+m)(l-m+1)}\Big[\langle nl(m-1)|v_0\rangle\, s_0\cdot e_- \\ &+\sqrt{(l-m)(l+m+1)}\,\langle nl(m+1)|v_0\rangle\, s_0\cdot e_+ \\ &+m\,\langle nlm|v_0\rangle\, s\cdot e_3\Big]\sin\omega_l t\Big\} \end{aligned} \tag{A5a}$$

$$\begin{aligned} p_1(v', t|v_0, s_0, 0) &\pm \mathrm{i}p_2(v', t|v_0, s_0, 0) \\ = \sum_{n,l,m}\langle v'|nlm\rangle \Big\{&\langle nml|v_0\rangle\, s_0\cdot e_\pm - \frac{1}{2l(l+1)}\sqrt{(l\pm m)(l\mp m+1)} \\ &\times\Big[\sqrt{(l\pm m-1)(l\mp m+2)}\,\langle nl(m\mp 2)|v_0\rangle\, s_0\cdot e_\mp \\ &+\sqrt{(l\pm m)(l\mp m+1)}\,\langle nlm|v_0\rangle\, s_0\cdot e_\pm\Big](1-\cos\omega_l t) \\ &+\frac{\mathrm{i}\nu_1}{\omega_l}\sqrt{(l\pm m)(l\mp m+1)}\,\langle nl(m\mp 1)|v_0\rangle\,\sin\omega_l t\Big\} \end{aligned} \tag{A5b}$$

$$\begin{aligned} p_3(v', t|v_0, s_0, 0) = \sum_{n,l,m}\langle v'|nlm\rangle \Big\{&\langle nml|v_0\rangle\, s_0\cdot e_3 \\ &-\frac{m}{2l(l+1)}\Big[\sqrt{(l+m)(l-m+1)}\,\langle nl(m-1)|v_0\rangle\, s_0\cdot e_- \\ &+\sqrt{(l-m)(l+m+1)}\,\langle nl(m+1)|v_0\rangle\, s_0\cdot e_+ \\ &+2m\langle nlm|v_0\rangle\, s_0\cdot e_3\Big](1-\cos\omega_l t) + \frac{i\nu_1}{\omega_l}m\,\langle nlm|v_0\rangle\,\sin\omega_l t\Big\}. \end{aligned} \tag{A5c}$$

In these formulae the following notation is introduced:

$$e_\pm = e_1 \pm \mathrm{i}e_2$$

$$\omega_l = \nu_1\sqrt{l(l+1)}$$

$$\langle v\,|nlm\rangle = \langle nlm|v\,\rangle^* = L_n^{l+1/2}\left(\frac{v^2}{2\bar{v}^2}\right)\left(\frac{v}{\sqrt{2}\bar{v}}\right)^l Y_{lm}\left(\frac{v}{v}\right)\left[\frac{n!}{\sqrt{2}\bar{v}^3}\Gamma^{-1}(n+l+3/2)\right]^{1/2}$$

the latter being the wavefunction of the 3D quantum harmonic oscillator.

The conditional probability (A2) can be used in the construction of a new kinetic model. By fixing the time in (A2) ($t = \tau \equiv$ constant) and by multiplying the Q-function by a collision rate ν, one obtains the function

$$A(v, s\,|v', s') = \nu\, Q(v, s, \tau|v', s', 0). \tag{A6}$$

Then one can use this function as a model collision kernel in the following kinetic equation:

$$\partial_t\, p(v, s, t) = -\nu\, p(v, s, t) + \int A(v, s\,|v', s')\, p(v', s', t)\ \mathrm{d}^3v'\, \mathrm{d}^2s' \qquad \text{(A7)}$$

where $p(v, s, t)$ is the P-symbol of the density matrix $\hat{\rho}(v, t)$ (Perelomov 1985). The model collision kernel should be considered as a generalized Keilson–Storer kernel. The ordinary Keilson–Storer collision kernel (see, e.g., McCourt *et al* 1990) has the following form:

$$A_{\mathrm{KS}}(v\,|v') \equiv \nu\, P_{\mathrm{OU}}(v, \tau|v', 0) \qquad \text{(A8)}$$

where $P_{\mathrm{OU}}(v, t|v', 0)$ is given by (A3). The Keilson–Storer collision kernel (A8) depends on two parameters: ν and $\nu_0\tau$, whereas the kernel (A6) depends on three parameters: ν, $\nu_0\tau$ and $\nu_1\tau$. The result of many weak collisions during the period $[0, \tau]$ appears as the effect of one collision in (A7). One can make sure that the kernel (A6) satisfies the main condition for any adequate model kernel, namely the detailed balance relation

$$A(v, s\,|v', s')\, W(v') = A(v', -s'\,|v, -s)\, W(v). \qquad \text{(A9)}$$

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